Signal Analysis Lectures

Lecture I: Detection filter

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Summary

1. Gravitational signal detection
2. Bayes Approach
   - Neyman Pearson rule
3. The Matched Filter
   - Signal Detection in a Gaussian noise
   - Signal to Noise Ratio

Gravitational signal detection

Defining the problem

- The problem of the detection and the problem of measure
- Signal analysis always provides probabilistic answers.
- The presence of noise in the detector creates problems in two ways: either simulate the presence of the signal or cover the presence of the signal
- The detection of a signal must be associated with a probability to express the level of confidence with which we are "certain" of the presence of the signal.
- Similarly, we must indicate a probability with which the measured parameters are in a certain range of values.

Signals and noise

Let's consider a $s(t)$ data sequence representing the output of the detector, $\boldsymbol{\theta}$ are the parameters of our signal

$$s(t) = \begin{cases} n(t) & \text{if the signal is not IN}\,, \\ n(t) + h(t, \boldsymbol{\theta}) & \text{if the signal is IN}\,. \end{cases} \tag{1}$$

Bayes Approach

Bayes Approach

- Let's assume that $\boldsymbol{\theta}$ is continuous.
- We indicate the probability that $h(t, \boldsymbol{\theta})$ is present in $s(t)$ with : $P(h|s) \equiv$ the probability that a signal $h(t, \boldsymbol{\theta})$, with $\boldsymbol{\theta}$ unknown is present in $s(t)$ sequence of data observed.

Notations

- $P(s|h)$ the probability to measure s assuming the signal h is present;
- $P(h)$ the apriori probability the signal h is present;
- $P(s)$ the probability the sequence of data $s(t)$ is observed;
- $P(0)$ the apriori probability the signal is not present ;
- $P(s|0)$ the probability density to observe $s(t)$ in absence of signal;
- $P[s|h(\boldsymbol{\theta})]$ the probability density to observe $s(t)$ assuming that $h(t, \boldsymbol{\theta})$ with a given $\boldsymbol{\theta}$ is present;
- $p(\boldsymbol{\theta})$ the apriori probability the signal $h(t)$ is characterized by $\boldsymbol{\theta}$.

The Bayes law

We can write the probability that a signal $h(t, \boldsymbol{\theta})$, with $\boldsymbol{\theta}$ unknown is present in $s(t)$ sequence of data observed.

$$P(h|s) = \frac{P(s|h)P(h)}{P(s)}\,. \tag{2}$$

Bayes Approach
Neyman Pearson rule

If we express $P(s)$ in terms of the two probabilities that the h signal is absent and that it is present, and also let's re-express the probability that h is present in terms of the probability that it is characterized by a particular $\boldsymbol{\theta}$ we have:

$$
\begin{aligned}
P(s) &= P(s|0)P(0) + P(s|h)P(h) && (3)\\
&= P(s|0)P(0) + P(h)\int d^N\theta\, p(\boldsymbol{\theta})P[s|h(\boldsymbol{\theta})] && (4)
\end{aligned}
$$

Using (2) and (3) we get:

$$P(h|s) = \frac{P(s|h)P(h)}{P(s|0)P(0) + P(h)\int d^N\theta\, p(\boldsymbol{\theta})P[s|h(\boldsymbol{\theta})} \tag{5}$$

We could write:

$$P(h|s) = \frac{\Lambda}{\Lambda + P(0)/P(h)}\,, \tag{6}$$

where

$$\Lambda \equiv \int d^N\theta\Lambda(\boldsymbol{\theta}) = \int d^N\theta p(\boldsymbol{\theta})\frac{P[s|h(\boldsymbol{\theta})]}{P(s|0)}\,. \tag{7}$$

Neyman Pearson rule

The conditional probability that the signal $h(t, \boldsymbol{\theta})$ with unknown parameters $\boldsymbol{\theta}$ is present in the observed data, depends on the apriori probability $P(0)$ and $P(h)$ and on the *likelihood ratio* Λ we have to estimate. Once we get $P(h|s)$, we have to choice a threshold to say if the signal is present or not. We can fix a threshold considering the False Alarm or the False Dismissal. If we do not know the apriori probabilities $P(h)$ and $P(0)$, we can fix the threshold not on $P(h|s)$, but on Λ:

$$\begin{array}{llll} \text{IF} & \Lambda \geq \Lambda_* & \text{The signal is present ;} \\ \text{IF} & \Lambda < \Lambda_* & \text{the signal is absent.} \end{array} \tag{8}$$

If we fix the threshold on Λ_* based on a given value of False Alarm we could accept, this is ***Neyman-Pearson decision rule***.

The Matched Filter

Signal Detection in a Gaussian noise

Signal Detection in a Gaussian noise

The conditional probability of measuring $s(t)$ if the particular signal $h(t, \boldsymbol{\theta})$ present is equal to the conditioned probability of measuring $s'(t) = s(t) - h(t, \boldsymbol{\theta})$. Assuming that the $h(t, \boldsymbol{\theta})$ signal is not present in $s'(t)$:

$$P[s|h(\boldsymbol{\theta})] = P[s - h(\boldsymbol{\theta})|0] \,. \tag{9}$$

Now let consider the probability to measure $s(t)$ assuming no signal inside $P(s|0)$ $s(t)$ is simple the noise $n(t)$.
Let suppose that $n(t)$ be a normal, zero mean, process with correlation function $C(\tau)$ (in Fourier domain with PSD $S(\nu)$).

To estimate $P[s|h(\boldsymbol{\theta})]/P(s|0)$ let's consider the continuous limit of discreetly sampled data $\{s_i : i = 1, \ldots, N\}$, with

$$s_i = s(t_i)\,, \tag{10}$$

$$t_i - t_j = (i-j)\Delta t\,, \tag{11}$$

$$\Delta t = \frac{T}{N-1}\,. \tag{12}$$

The probability that a particular s is a sample of the process random $n(t)$ given by

$$P(s_i|0) = \frac{\exp\left[-\frac{1}{2}\frac{s_i^2}{C(0)}\right]}{[2\pi C(0)]^{1/2}}\,, \tag{13}$$

The probability that the ordered set $\{s_i : i = 1, \ldots, N\}$ is a sampling of $n(t)$ is

$$P(s|0) = \frac{\exp\left[-\frac{1}{2}\sum_{j,k=1}^{N} C_{jk}^{-1} s_j s_k\right]}{(2\pi)^N \det[C_{ij}]^{1/2}}, \tag{14}$$

where

$$C_{ij} \equiv C[(i-j)\Delta t]. \tag{15}$$

exploiting previous relationships and the Wiener-Khintchine theorem, prove that

$$e^{2\pi i \nu t_k} = \lim_{\Delta t \to 0} \frac{1}{\Delta t^2} \frac{1}{2} S(\nu) \widetilde{C^{-1}}(\nu, t_k), \tag{16}$$

where

$$\widetilde{C^{-1}}(\nu, t_k) \equiv \int_{-\infty}^{\infty} dt C^{-1}(t, t_k) e^{2\pi i \nu t}. \tag{17}$$

with $\widetilde{C^{-1}}$ ed the Parsival theorem we can write

$$\lim_{\substack{\Delta t \to 0 \\ T \to \infty}} \sum_{j,k=1}^{N} C_{jk}^{-1} s_j s_k = \int_{-\infty}^{\infty} d\nu \frac{\tilde{s}(\nu)\tilde{s}^*(\nu)}{S(\nu)} \tag{18}$$

We introduced the symmetric scalar internal product

$$< s, h > \equiv \int_{-\infty}^{\infty} d\nu \frac{\widetilde{s}(\nu)\widetilde{h^*}(\nu)}{S(\nu)} \tag{19}$$

for the real function s e h.
So, we can write the *likelihood*

$$\begin{aligned} \Lambda(\boldsymbol{\theta}) = p(\boldsymbol{\theta})\frac{P[s|h(\boldsymbol{\theta})]}{P(s|0)} &= p(\boldsymbol{\theta})\frac{P[s-h(\boldsymbol{\theta})|0]}{P(s|0)} \\ &= p(\boldsymbol{\theta})\frac{\exp\left[-\frac{1}{2} < s-h(\boldsymbol{\theta}), s-h(\boldsymbol{\theta}) >\right]}{\exp\left[-\frac{1}{2} < s, s >\right]} \\ &= p(\boldsymbol{\theta})\exp\left[< s, h(\boldsymbol{\theta}) > -\frac{1}{2} < h(\boldsymbol{\theta}), h(\boldsymbol{\theta}) >\right]. \end{aligned} \tag{20}$$

At this point the study of the optimal detector depends on each individual case by the amount of information we have on the parameters $\boldsymbol{\theta}$.

The Matched Filter
Signal to Noise Ratio

Signal to Noise Ratio

Let's consider now the simple detection of a signal with known waveform and parameters in a stationary gaussian background noise.
The *likelihood* ratio is

$$\Lambda = \exp\left(< s, h > - \frac{1}{2} < h, h > \right) , \tag{21}$$

where only $< s, h >$ depends on s and Λ a Monotonic growing function of $< s, h >$. So, we can use $< s, h >$ as decision rule on a threshold e s_*:

$$\begin{array}{llll} \text{IF} & < s, h > & \geq & s_* \quad \text{the signal is present,} \\ \text{IF} & < s, h > & < & s_* \quad \text{the signal is absent.} \end{array} \tag{22}$$

The operator $< s, h >$ is linear in s s; this means that the optimal detector in presence of Gaussian noise is a detector which depends only on the signal shape h and noise power spectral density $S(\nu)$.

The action of a linear filter for signals of known shape may be characterized by the **signal-to-noise** ratio at the filter output.

$$\rho = \frac{< h, h >^2}{< h, n >^2} =< h, h >\equiv \int_{-\infty}^{\infty} d\nu \frac{\widetilde{h}(\nu)\widetilde{h^*}(\nu)}{S(\nu)} \ . \tag{23}$$

The probability of false alarms and correct detection are determined by the amount of s and the density of $< s, h >$. The amount of $< s, h >$, being a linear combination of the variables Gaussian s_i is itself Gaussian distributed. If we receive only noise, then

$$\mathcal{E}[< n, h >] = 0 \qquad \text{e} \qquad \mathcal{E}[< n, h >^2] =< h, h > \ , \tag{24}$$

where $\mathcal{E}[\ ]$ is the expectation value. If we have $s = h + n$, then

$$\mathcal{E}[< s, h >] =< h, h > \qquad \text{e} \qquad \mathcal{E}[< s, h >^2] =< h, h > \ . \tag{25}$$

Without any signal the probability density of $< s, h >$ is

$$p_0(< s, h >) = \frac{1}{\sqrt{2\pi < h, h >}} \exp - \frac{< s, h >^2}{2 < h, h >} \ , \qquad (26)$$

while with signal is

$$p_h(< s, h >) = \frac{1}{\sqrt{2\pi < h, h >}} \exp - \frac{(< s, h > - < h, h >)^2}{2 < h, h >} \ . \qquad (27)$$

The False alarm probability F is equal to the probability that $< s, h >$ is greater than s_* in presence of only noise:

$$F = \int_{s_*}^{\infty} p_0(< s, h >) d < s, h >= \frac{1}{\sqrt{2\pi}} \int_{\phi_*}^{\infty} \exp\left\{ -\frac{\phi^2}{2} d\phi \right\} \ , \qquad (28)$$

where $\phi_* = s_* / < h, h >$.

In a similar way, the probability of correct detection is

$$D = \int_{s_*}^{\infty} p_h(< s, h >) d < s, h >= \frac{1}{\sqrt{2\pi}} \int_{\phi'_*}^{\infty} \exp\left\{ -\frac{\phi^2}{2} d\phi \right\}, \qquad (29)$$

where $\phi'_* = \phi_* - \sqrt{< h, h >}$. If we fix F and D we can find which is the minimum SNR ρ:

$$\rho =< h, h >= (\phi_* - \phi'_*)^2 \qquad \text{per} \qquad D \geq F\,. \qquad (30)$$

The Matched or Wiener Filter

The matched or Wiener filter is given by

$$\rho = \int_{-\infty}^{\infty} d\nu \frac{\widetilde{s}(\nu)\widetilde{h^*}(\nu)}{S(\nu)} \ . \tag{31}$$

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